

Fault-tolerant, Universal Adiabatic Quantum Computation

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Abstract

Quantum computation has revolutionary potential for speeding computational tasks such as factoring and simulating quantum systems, but the task of constructing a quantum computer is daunting. Adiabatic quantum computation and other “hands-off” approaches relieve the need for rapid, precise pulsing to control the system, inspiring at least one high-profile effort to realize a hands-off quantum computing device. But is hands-off incompatible with fault-tolerant? Concerted effort and many innovative ideas have not resolved this question but have instead deepened it, linking it to fundamental problems in quantum complexity theory. Here we present a hands-off approach that is provably (a) capable of scalable universal quantum computation in a non-degenerate ground state and (b) fault-tolerant against an analogue of the usual local stochastic fault model. A satisfying physical and numerical argument indicates that (c) it is also fault-tolerant against thermal excitation below a threshold temperature independent of the computation size.

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The discovery of quantum algorithms [1, 2] and quantum error correction [3] ushered in a period of intense interest in quantum computing. The standard gate approach [4] to realizing a quantum computer involves fashioning a collection of two state quantum systems – qubits – that can be individually manipulated by time-dependent gates. The experimental requirements of the standard gate approach are intimidating enough that alternative approaches have garnered interest. One important class relaxes the need for vigorous, precisely-timed pulsing of the Hamiltonian, requiring only a time-independent or an adiabatically-changing Hamiltonian [5–13]. It is known that such “hands-off” approaches permit arbitrary quantum computation in a zero-temperature, error-free circumstance. However, sustained and clever theoretical attack [14–17] has not resolved whether such approaches permit scalable, fault-tolerant computation in noisy environments; indeed, there have been some reasons for pessimism [17–20]. This problem is pressing given that high-profile efforts to fabricate such systems are already underway [21]; is any such effort doomed, even in principle, from scaling up? The question also turns out to have deep connections to quantum complexity theory [22].

Here, we formulate a hands-off approach to quantum computation that permits universal computation in a non-degenerate ground state and is provably fault-tolerant against “local stochastic excitations.” In addition, we supply a satisfying numerical and physical argument that the approach is fault-tolerant against thermal excitations. Our formulation involves a Hamiltonian of 2-body interactions characterized by an energy scale ϵ that is independent of system size. The approach is based upon the formalism of ground-state quantum computation (GSQC) [8–10, 12] that replaces the time-dependent quantum state with a “history state” $|\phi\rangle$ in a larger Hilbert space [23]. This history state possesses a time-independent record of the entire evolution of the time-dependent state. GSQC works by formulating a Hamiltonian whose ground state is the history state of a given circuit. By cooling or by adiabatically tuning the Hamiltonian [9], one imagines bringing the system into its ground state and then measuring the results of the computation. Unfortunately, as Hastings has pointed out, a Lieb-Robinson argument [24] essentially ensures the GSQC ground state will be vulnerable to excitation. The key innovation in the following is the leveraging of fault-tolerance in the standard gate model to ensure that *excited* states of the Hamiltonian are computationally

meaningful in addition to the ground state.

Consider a quantum circuit C , expressed in the gate approach as a sequence of G_1 one-qubit and G_2 two-qubit unitary gates U_i . C has Q physical qubits, each of which is initialized to state $|0\rangle$. Assume that C involves no measurements and that C is fault-tolerant [25] against a local stochastic fault model [26]. To frame this model, define a fault-path $F(\ell)$ as a function of each location ℓ in the circuit such that $F(\ell) = 0$ if there is no fault at location ℓ . If there is a fault at location ℓ , $F(\ell)$ lies within a range of possible fault types $\{1, \dots, f_{max}(\ell)\}$. The model stipulates that a fault path, chosen at random, incorporates faults at a specific set of L locations in C with probability no greater than \bar{p}^L , for some \bar{p} strictly less than 1. Let $|\phi\rangle$ be the final state of the Q qubits in the ideal case in which no faults occur during the execution of C . Because C encodes physical qubits into logical qubits, we can extract the correct answer even if the final state is, instead of $|\phi\rangle$, a correctable state with not too many error operators applied to $|\phi\rangle$. Let the “result” operator R be the projector given by summing the dyad $|\phi\rangle\langle\phi|$ and a dyad for each of the correctable states. Let ρ_R be the density matrix of the Q qubits that results from the execution of C . The fault-tolerance of C implies that $\text{Tr } R \rho_R / \text{Tr } \rho_R \sim O(1)$ if \bar{p} is much less than a probability p .

Having characterized C , we describe how to construct its history-state equivalent. We supply an explicit map to a Hamiltonian $\mathcal{H}(\theta)$ defined on a $2^{\otimes Q} \otimes (3 \times 5)^{\otimes G_1 + 2G_2}$ dimensional Hilbert space. The Hamiltonian is a sum of initialization terms involving $H_{Initialize}$, one-qubit gate terms involving $H_{One-qubit\ gate}^{U_i}(\theta)$, and two-qubit gate terms $H_{Two-qubit\ gate}^{U_j}(\theta)$ lying in one-to-one correspondence to the steps of C . To execute an adiabatic computation with $\mathcal{H}(\theta)$, one takes θ from 0 to a value Θ close to $\pi/2$ in a time that scales with $G = G_1 + G_2$. The answer to the computation is contained on the $2^{\otimes Q}$ part of the Hilbert space, after tracing out the $(3 \times 5)^{\otimes G_1 + 2G_2}$ part.

To describe the map, for each gate approach circuit of Fig. 1, we give the corresponding history-state Hamiltonian $\mathcal{H}(\theta)$ in Fig. 2.

(i) To initialize Q qubits, define a 2 dimensional Hilbert space for each qubit, so that the full space has dimension $2^{\otimes Q}$. Each qubit has a basis $\{|0_0\rangle, |1_0\rangle\}$ where the ket $|b_s\rangle$ has “bit” value b and computational “stage” value s . The Hamiltonian is $\mathcal{H} = \mathcal{H}_{Initialize} = \sum_q I^{\otimes q-1} \otimes H_{Initialize} \otimes I^{\otimes Q-q}$ where $H_{Initialize} = \epsilon |1_0\rangle\langle 1_0|$, I denotes the identity operator, and ϵ is a fixed energy scale.

The non-degenerate ground state $|\Psi\rangle = |0_0\rangle^{\otimes Q}$ is the lone zero-energy eigenstate of the positive semi-definite \mathcal{H} : $\mathcal{H}|\Psi\rangle = 0$.

(ii) To apply a single-qubit gate U_1 to qubit q after initialization, extend the $2^{\otimes Q} = 2^{\otimes q-1} \otimes 2 \otimes 2^{\otimes Q-q}$ dimensional space to a $2^{\otimes q-1} \otimes 4 \otimes 2^{\otimes Q-q}$ dimensional space. Supplement the basis $\{|0_0\rangle, |1_0\rangle\}$ of qubit q so that there are 4 basis states $\{|0_0\rangle, |1_0\rangle, |0_1\rangle, |1_1\rangle\}$. To incorporate the effect of U_1 , let $|\Psi\rangle = |0_0\rangle^{\otimes q-1} \otimes (|0_0\rangle + |0_1\rangle \langle 0| U_1 |0\rangle + |1_1\rangle \langle 1| U_1 |0\rangle) / \sqrt{2} \otimes |0_0\rangle^{\otimes Q}$. This $|\Psi\rangle$ deserves the name “history state” because it is comprised of a superposition of the initialized qubit $|0_0\rangle$ corresponding to stage 0 before U_1 acts and $|0_1\rangle \langle 0| U_1 |0\rangle + |1_1\rangle \langle 1| U_1 |0\rangle$ corresponding to stage 1 after U_1 acts. The subscript $s = 0, 1$ in the ket $|b_s\rangle$ keeps track of the stage. If we carry U_1 over to the extended Hilbert space by defining the operator $\mathcal{U}_1 = \sum_{b,b',s=0,1} |b_s\rangle \langle b| U_1 |b'\rangle \langle b'_s|$, then $|\Psi\rangle = |0_0\rangle^{\otimes q-1} \otimes (|0_0\rangle + \mathcal{U}_1 |0_1\rangle) / \sqrt{2} \otimes |0_0\rangle^{\otimes Q}$. A positive semi-definite Hamiltonian that satisfies $\mathcal{H}|\Psi\rangle = 0$ is $\mathcal{H} = \mathcal{H}_{Initialize} + I^{\otimes q-1} \otimes H^{\mathcal{U}_1} \otimes I^{\otimes Q-q}$ where

$$H^{\mathcal{U}_1} = \epsilon \sum_b (|b_0\rangle - \mathcal{U}_1 |b_1\rangle)(\langle b_0| - \langle b_1| \mathcal{U}_1^\dagger) / 2. \quad (1)$$

To achieve a fault-tolerant construction, we will incorporate a teleportation-like step [27] after the action of U_1 (Fig. A1). Extend the space of qubit q again by direct sum with another state to form a 5 dimensional Hilbert space and then further extend by direct product with a 3 dimensional space and with a 2 dimensional space. The space of qubit q has gone from 2 to 4 and now to $2 \otimes 3 \otimes 5$ dimensional. A convenient basis is $\{|0_0\rangle, |1_0\rangle\} \otimes \{|0_0\rangle, |1_0\rangle, |\text{IDLE}\rangle\} \otimes \{|0_0\rangle, |1_0\rangle, |0_1\rangle, |1_1\rangle, |\text{IDLE}\rangle\}$. The history state is assigned the form $|\Psi\rangle = |0_0\rangle^{\otimes q-1} \otimes |\psi_0^{U_1}(0)\rangle \otimes |0_0\rangle^{\otimes Q-q}$ where

$$\begin{aligned} |\psi_0^{U_1}(b)\rangle &= [\sqrt{2} \cos \theta (|0_0\rangle \otimes |0_0\rangle + |1_0\rangle \otimes |1_0\rangle) \otimes (|b_0\rangle + \mathcal{U}_1 |b_1\rangle) \\ &\quad + \sin \theta \mathcal{U}_1 |b_0\rangle \otimes |\text{IDLE}\rangle \otimes |\text{IDLE}\rangle] / \sqrt{8 \cos^2 \theta + \sin^2 \theta}. \end{aligned} \quad (2)$$

The first part of $|\psi_0^{U_1}(b)\rangle$ prepends, alongside our former history state $(|b_0\rangle + \mathcal{U}_1 |b_1\rangle) / \sqrt{2}$, a Bell-pair that will be used for teleportation. The second part $\mathcal{U}_1 |b_0\rangle \otimes |\text{IDLE}\rangle \otimes |\text{IDLE}\rangle$ mimics the post Bell-basis measurement step of teleportation, consuming the original qubit state and half of the Bell-pair, and teleporting the original qubit state to the other half of the Bell-pair. One finds that $\mathcal{H}|\Psi\rangle = 0$ for the positive semi-definite Hamiltonian $\mathcal{H}(\theta) = \mathcal{H}_{Initialize} + I^{\otimes q-1} \otimes H_{One-qubit\ gate}^{U_1} \otimes$

$I^{\otimes Q-q}$ where $H_{One-qubit\ gate}^{U_1} = I \otimes I \otimes H^{\mathcal{U}_1} + H_{Create\ pair} \otimes I + I \otimes H_{Bell\ projection}(\theta)$. The $I \otimes I \otimes H^{\mathcal{U}_1}$ term is defined by (1). The second term contains

$$\begin{aligned} H_{Create\ pair} = \frac{\epsilon}{2} [& (|1_0\rangle |0_0\rangle - |0_0\rangle |1_0\rangle)(\langle 1_0| \langle 0_0| - \langle 0_0| \langle 1_0|) \\ & + (|1_0\rangle |0_0\rangle + |0_0\rangle |1_0\rangle)(\langle 1_0| \langle 0_0| + \langle 0_0| \langle 1_0|) \\ & + (|0_0\rangle |0_0\rangle - |1_0\rangle |1_0\rangle)(\langle 0_0| \langle 0_0| - \langle 1_0| \langle 1_0|)] \end{aligned} \quad (3)$$

to impose an energy penalty if the Bell-pair in the first part of $|\psi_0^{U_1}(b)\rangle$ is not of the desired form $(|0_0\rangle \otimes |0_0\rangle + |0_0\rangle \otimes |0_0\rangle)/\sqrt{2}$. The third term

$$\begin{aligned} H_{Bell\ projection}(\theta) = \epsilon |\text{IDLE}\rangle \langle \text{IDLE}| \otimes \sum_{b,s=0,1} |b_s\rangle \langle b_s| + \epsilon \sum_{b=0,1} |b_0\rangle \langle b_0| \otimes |\text{IDLE}\rangle \langle \text{IDLE}| \quad (4) \\ + \epsilon \left(\sin \theta \frac{|0_0\rangle |0_1\rangle + |1_0\rangle |1_1\rangle}{\sqrt{2}} - \cos \theta |\text{IDLE}\rangle \langle \text{IDLE}| \right) \\ \left(\sin \theta \frac{\langle 0_0| \langle 0_1| + \langle 1_0| \langle 1_1|}{\sqrt{2}} - \cos \theta \langle \text{IDLE}| \langle \text{IDLE}| \right), \end{aligned}$$

mimics the effects of the Bell-basis measurement step of teleportation and imposes an energy penalty unless both targets of the measurement undergo the step in tandem.

Given that qubit q now has a $2 \otimes 3 \otimes 5$ dimensional space, in what sense is it still a qubit? If we compute the density matrix of the system and trace out the $3 \otimes 5$ part, the remaining 2 dimensional space contains its quantum information. To make this clear, define the "gate operator" $\hat{g}_0^{U_1} = |\psi_0^{U_1}(0)\rangle \langle 0_0| + |\psi_0^{U_1}(1)\rangle \langle 1_0|$ in terms of (2). This operator is a mapping from a 2 dimensional space to a $2 \otimes 3 \otimes 5$ dimensional space. In terms of this definition, the history state is $|\Psi\rangle = |0_0\rangle^{\otimes q-1} \otimes \hat{g}_0^{U_1} |0_0\rangle \otimes |0_0\rangle^{\otimes Q-q}$. Define the superoperator $g_0^{U_1}(\rho) = \text{Tr}_{3 \otimes 5} \hat{g}_0^{U_1} \rho \hat{g}_0^{U_1 \dagger}$. We find that $g_0^{U_1}(\rho)$ applies U_1 and then a depolarizing channel of probability $p_0^{U_1} = (8 \cos^2 \theta)/(8 \cos^2 \theta + \sin^2 \theta)$. If one were to measure the final 2 dimensional part of the $2 \otimes 3 \otimes 5$ dimensional space of the qubit, the density matrix $\text{Tr}_{3 \otimes 5} |\Psi\rangle \langle \Psi| = (|0_0\rangle \langle 0_0|)^{\otimes q-1} \otimes g_0^{U_1}(|0_0\rangle \langle 0_0|) \otimes (|0_0\rangle \langle 0_0|)^{\otimes Q-q}$ would determine the result.

(iii) Suppose that there is a second single-qubit gate U_2 acting on qubit q' . If $q' > q$, we set the system to dimension $2^{\otimes q-1} \otimes (2 \otimes 3 \otimes 5) \otimes 2^{\otimes q'-q-1} \otimes (2 \otimes 3 \otimes 5) \otimes 2^{\otimes Q-q'}$. The history state is $|\Psi\rangle = |0_0\rangle^{\otimes q-1} \otimes \hat{g}_0^{U_1} |0_0\rangle \otimes |0_0\rangle^{\otimes q'-q-1} \otimes \hat{g}_0^{U_2} |0_0\rangle \otimes |0_0\rangle^{\otimes Q-q'}$. The Hamiltonian

is $\mathcal{H}(\theta) = \mathcal{H}_{Initialize} + I^{\otimes q-1} \otimes H_{One-qubit\ gate}^{U_1} \otimes I^{\otimes Q-q} + I^{\otimes q'-1} \otimes H_{One-qubit\ gate}^{U_2} \otimes I^{\otimes Q-q'}$.

The density matrix over the final 2 dimensional Hilbert space of every qubit is $(|0_0\rangle\langle 0_0|)^{\otimes q-1} \otimes g_0^{U_1}(|0_0\rangle\langle 0_0|) \otimes (|0_0\rangle\langle 0_0|)^{\otimes q'-q-1} \otimes g_0^{U_2}(|0_0\rangle\langle 0_0|) \otimes (|0_0\rangle\langle 0_0|)^{\otimes Q-q'}$.

If $q' = q$, iterate step (ii). Extend the 2 dimensional part of qubit q 's Hilbert space to $2 \otimes 3 \otimes 5$ dimensional so the space of the system goes from $2^{\otimes q-1} \otimes (2 \otimes 3 \otimes 5) \otimes 2^{\otimes Q-q}$ dimensional to $2^{\otimes q-1} \otimes (2 \otimes 3 \otimes 5 \otimes 3 \otimes 5) \otimes 2^{\otimes Q-q}$ dimensional. The Hamiltonian becomes $\mathcal{H}(\theta) = \mathcal{H}_{Initialize} + I^{\otimes q-1} \otimes (I \otimes I \otimes H_{One-qubit\ gate}^{U_1}) \otimes I^{\otimes Q-q} + I^{\otimes q-1} \otimes (H_{One-qubit\ gate}^{U_2} \otimes I \otimes I) \otimes I^{\otimes Q-q}$. The history state is $|\Psi\rangle = |0_0\rangle^{\otimes q-1} \otimes \hat{g}_0^{U_2} \hat{g}_0^{U_1} |0_0\rangle \otimes |0_0\rangle^{\otimes Q-q}$. The density matrix over the final 2 dimensional Hilbert space of every qubit is $(|0_0\rangle\langle 0_0|)^{\otimes q-1} \otimes g_0^{U_2} \left(g_0^{U_1} (|0_0\rangle\langle 0_0|) \right) \otimes (|0_0\rangle\langle 0_0|)^{\otimes Q-q}$

(iv) We incorporate a two-qubit gate U_3 . For concreteness, assume qubits with adjacent labels q'' and $q'' + 1$ are undergoing the gate. Extend the final 2 dimensional space of each qubit to $2 \otimes 3 \otimes 5$ dimensional. Add a term $I \otimes \dots \otimes I \otimes H_{Two-qubit\ gate}^{U_3} \otimes I \otimes \dots \otimes I$ to the Hamiltonian, consisting of a two-qubit version of (1) followed by teleportation Hamiltonians (3) and (4) acting on each qubit. The two-qubit states $|\Psi_0^{U_3}(b, B)\rangle$ that satisfy $H_{Two-qubit\ gate}^{U_3} |\Psi_0^{U_3}(b, B)\rangle = 0$ are analogous to (2). Use them to define a gate operator $\hat{g}_0^{U_3} = \sum_{b,B} |\Psi_0^{U_3}(b, B)\rangle \langle b_0| \otimes \langle B_0|$ that maps a $2 \otimes 2$ dimensional space to a $(2 \otimes 3 \otimes 5) \otimes (2 \otimes 3 \otimes 5)$ dimensional space; the associated superoperator is $g_0^{U_3}(\rho) = \text{Tr}_{3 \otimes 5} \text{Tr}_{3 \otimes 5} \hat{g}_0^{U_3} \rho \hat{g}_0^{U_3 \dagger}$. We find that $g_0^{U_3}(\rho)$ applies the desired gate with probability $1 - p_0^{U_3}$ and introduces an error into the output density matrix with probability $p_0^{U_3} = (32 \cos^4 \theta + 8 \cos^2 \theta \sin^2 \theta)(32 \cos^4 \theta + 8 \cos^2 \theta \sin^2 \theta + \sin^4 \theta)$.

By iterating the constructions above for all of the gates in the circuit **C**, one obtains the Hilbert space and Hamiltonian of a history-state simulation of the complete circuit. The Hamiltonian is positive semi-definite and has a non-degenerate ground state $|\Psi\rangle$ of energy 0. In terms of the $\hat{g}_0^{U_i}$ gate operators, $|\Psi\rangle = (I^{\otimes j} \otimes \hat{g}_0^{U_G} \otimes I^{\otimes j'}) \dots (I^{\otimes j''} \otimes \hat{g}_0^{U_1} \otimes I^{\otimes j'''}) |0_0\rangle^{\otimes Q}$ for some values of j, j', j'' , and j''' that depend on the sequence of gates of the circuit. No confusion should arise if we suppress this cumbersome notation and write $|\Psi\rangle = \hat{g}_0^{U_G} \dots \hat{g}_0^{U_1} |0_0\rangle^{\otimes Q}$. The final result density matrix of dimension $2^{\otimes Q}$, obtained by tracing out all but the final 2 dimensional Hilbert space of each qubit, is $\rho_R = g_0^{U_G} (\dots g_0^{U_1} (|0\rangle\langle 0|^{\otimes Q}) \dots)$. This equals the density matrix that would be produced by executing **C** in the gate approach with depolarization fault rate $p_0^{U_i}$ on gate i .

C's fault-tolerance implies $\text{Tr } R \rho_R / \text{Tr } \rho_R \sim O(1)$ provided the largest gate fault rate, $p_0 = (32 \cos^4 \theta + 8 \cos^2 \theta \sin^2 \theta)(32 \cos^4 \theta + 8 \cos^2 \theta \sin^2 \theta + \sin^4 \theta) \ll p$. This proves that the history-state $|\Psi\rangle$ can be used to perform universal quantum computation, outputting the correct answer if θ is sufficiently close to $\pi/2$.

The history-state construction is moreover fault-tolerant with respect to a “local stochastic excitation” model. This is the natural history-state analogue of the local stochastic fault model described above; we borrow the same definition of the fault path $F(\ell)$ and the same probability restriction. The new aspect is that, if $F(\ell) = 0$ for all locations, then the system is assumed to occupy a state that is annihilated by every term $I \otimes \dots \otimes I \otimes H_{\text{Initialize}} \otimes I \otimes \dots \otimes I$, $I \otimes \dots \otimes I \otimes H_{\text{One-qubit gate}}^{U_i} \otimes I \otimes \dots \otimes I$, and $I \otimes \dots \otimes I \otimes H_{\text{Two-qubit gate}}^{U_j} \otimes I \otimes \dots \otimes I$ in the history-state Hamiltonian $\mathcal{H}(\theta)$. In other words, the system occupies our history-state $|\Psi\rangle$, the solution to $\mathcal{H}(\theta) |\Psi\rangle = 0$. If $F(\ell) \neq 0$ at some locations, the system is assumed to occupy a different time-independent state $|\Psi_F\rangle$ that is annihilated by all terms in $\mathcal{H}(\theta)$ except for terms corresponding to the faulty locations. We refer to $|\Psi_F\rangle$ as a “locally excited” state; in general it is not an eigenstate of $\mathcal{H}(\theta)$.

Suppose our history-state version of C, under the local stochastic excitation model, produces some ρ_R . One can prove that the same ρ_R could have been obtained by executing the gate approach circuit C within a non-Markovian fault model with a shared bath formed by the tensor product of one $3 \otimes 5$ dimensional space per faulty location. Because of the fault-tolerance of the gate approach circuit C with respect to such a non-Markovian fault model [25], it follows that $\text{Tr } R \rho_R / \text{Tr } \rho_R \sim O(1)$ provided $p_0 + \bar{p} \ll p$.

Taking a different perspective, we have proven fault-tolerance against a model of fabrication faults in which the actual Hamiltonian \mathcal{H}' differs from $\mathcal{H}(\theta)$ at faulty locations $F(\ell) \neq 0$.

Turning from the stochastic local excitation model to a generic thermal excitation model, imagine constructing our history-state Hamiltonian $\mathcal{H}(\theta)$ and bringing it into thermal equilibrium with a bath at temperature T . Our quantity of interest is then a thermal average[28]; we write $\text{Tr } R \rho_R / \text{Tr } \rho_R = \text{Tr } R e^{-\beta \mathcal{H}} / \text{Tr } e^{-\beta \mathcal{H}}$.

We expect the system to be gapless [24]. Numerical simulations provide a lucid picture of

the excited states. Consider first a one dimensional chain of one-qubit gates (Fig. 3A). Divide the chain into unit cells each consisting of one gate. Guess a variational form for the energy eigenstates, $|\text{one gate}\rangle = \sum_j e^{ikj} \dots \hat{g}_0^{U_{j+1}} \hat{g}_f^{U_j} \hat{g}_0^{U_{j-1}} \dots |0\rangle$ in terms of a faulty gate operator $\hat{g}_f^{U_j} = \sum_b |\psi_f^{U_j}(b)\rangle \langle b|$. Regarding \mathcal{H} as a spin Hamiltonian, $|\text{one gate}\rangle$ is a spin wave. Minimizing the energy leads to a coupled linear equation for $|\psi_f^{U_j}(0)\rangle$ and $|\psi_f^{U_j}(1)\rangle$. After solving the equation numerically, reanalyze the same one dimensional chain, this time thinking of the unit cell as 4 one-qubit gates. This larger simulation is much less constrained: periodic structure from unit cell to unit cell is still imposed by our variational guess, but the state can assume any form within the unit cell of 4 gates. Strikingly, when we minimize its energy and solve the resulting equations, the output turns out to be very close to $|\text{one gate}\rangle$. Fig. 4A demonstrates that the infidelity over the 4 gate unit cell is exceedingly small for any value of θ . We perform similar exercises on 3 more configurations (Fig. 3). The small infidelity found in all cases (Fig. 4) makes an extremely strong case for spin-wave excitations. Since these excited states do not involve large domains of errors, we do not expect them to overwhelm the fault-tolerance of **C** provided their density of states is sufficiently small.

Generalizing, we write the low-lying excited states of $\mathcal{H}(\theta)$ as spin waves $\sum_F \chi(F) |\Psi_F\rangle$. A type f spin wave ($f = 1, \dots, f_{max}$) has energy $E(k, f)$, where k denotes some set of quantum numbers. Treating the spin waves as non-interacting bosons to first approximation [29, 30] enables one to write down a state $|n(k, f)\rangle = \sum_F \chi_{n(k, f)}(F) |\Psi_F\rangle$ with a distribution $n(k, f)$ of spin waves and energy $\sum_{k, f} E(k, f) n(k, f)$. In thermal equilibrium, the bosons produce faults at L given locations with probability less than \bar{p}^L for $\bar{p} \approx p(T) = \sum_{k, f} \langle n(k, f) \rangle / G + Q$. Here, $\langle n(k, f) \rangle = 1/(e^{\beta E(k, f)} - 1)$ is the average boson occupancy. Including the faults that arise even in the ground state of \mathcal{H} , we conclude $F(\ell)$ satisfies the local stochastic excitation model with $\bar{p} = p_0 + p(T)$. Thus, $\text{Tr } R \rho_R / \text{Tr } \rho_R \sim O(1)$ if $p_0 + p(T) \ll p$.

Hopping from gate to gate, the traveling $\hat{g}_f^{U_j}$ within a spin wave produces a quadratic dispersion $E(k, f) \sim \epsilon k^2 / \mu(\theta, f)$ for low-lying f and θ near $\pi/2$ (Fig. A2). If the gate approach circuit **C** is 2 dimensional in space, its history-state version is 3 dimensional in space, and the density of states is low enough for $p(T)$ to converge; the threshold condition becomes $p_0(\theta) +$

$\sum_f (\mu(\theta, f) k_B T / \epsilon)^{3/2} \ll p$. Choosing a large C and taking $\theta = \Theta \equiv \pi/2 - (p/8C)^{1/4}$, the history-state produces reliable output if $k_B T \leq k_B T_{\text{threshold}} \approx \epsilon(p/C)^{2/3} / \sum_f \mu(\pi/2 - (p/8C)^{1/4}, f)$.

The time to compute is the period it takes to bring $\mathcal{H}(\Theta)$ into thermal equilibrium at temperature T . Imagine starting $\mathcal{H}(\theta)$ at $\theta = 0$; here the system is comprised of isolated subsystems that should reach equilibrium quickly. How fast can one increase θ from 0 to Θ while maintaining thermal equilibrium?

Consider a generic model of dissipation. Suppose the system is coupled to a bath with a Hamiltonian $H_{SB} = \sum_{\alpha} A_{\alpha} \otimes B_{\alpha}$. Here A_{α} is a projector on to a basis state of the system, like $(I^{\otimes j} \otimes |0\rangle\langle 0| \otimes I^{\otimes j'})$. The operator $B_{\alpha} = \sum_n k_{n,\alpha} x_{n,\alpha}$ acts on coordinates $x_{n,\alpha}$ of the bath; these coordinates appear within the bath Hamiltonian as $H_B = \sum_{n,\alpha} (1/2 m_{n,\alpha}) p_{n,\alpha}^2 + (m_{n,\alpha} \omega_{n,\alpha}^2 / 2) x_{n,\alpha}^2$. For a given A_{α} , matrix elements between the spin wave eigenstates of the Hamiltonian scale as $1/G$. Within standard masters equations methods [31], relaxation rates scale with the matrix element squared, or $1/G^2$ in our case. Summing over the number of values of α , which scales as G , we find a net relaxation rate that scales as $G(1/G^2) = 1/G$. For $d\theta/dt$ much less than these relaxation rates, thermal equilibrium should be maintained, so we take $d\theta/dt \sim c/G$ for some small c . The associated time to compute is linear in G . Note that, in contrast to typical adiabatic conditions, the system gap does not enter this argument; for large systems it is likely to be much smaller than the temperature, so the system will have negligible probability of being in the ground state irrespective of the value of $d\theta/dt$.

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authors of [11]. We sometimes find it helpful to describe the history state as quantum graph paper on which the time-dependent trajectory is recorded: $|\phi(t)\rangle$ versus t .

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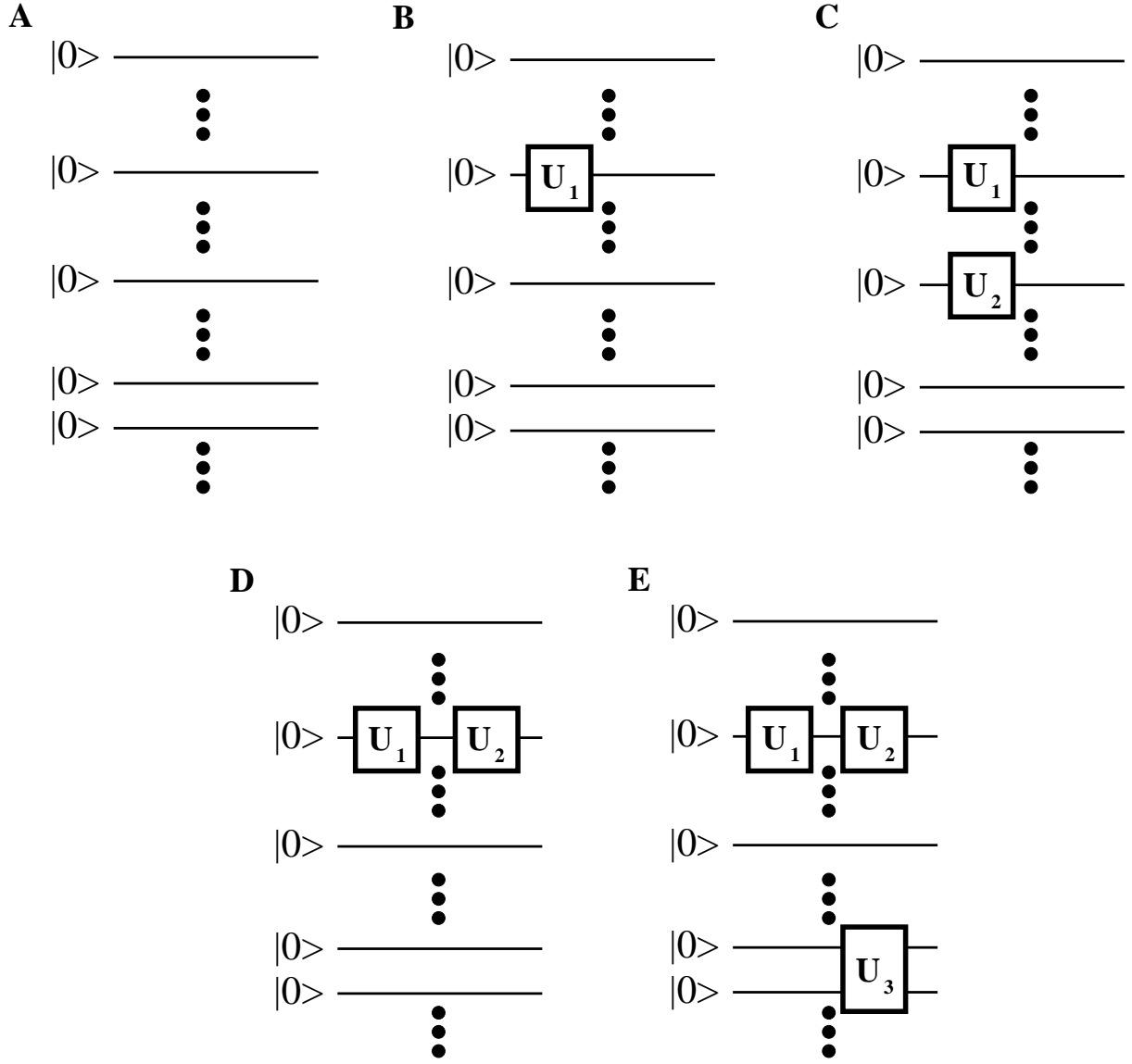


FIG. 1. Gate approach circuits used to illustrate the history-state construction. (A) Initialization of qubits. (B) Initialization of qubits, followed by a one-qubit gate on qubit q . (C) Circuit (B) followed by a one-qubit gate on qubit $q' > q$. (D) Circuit (B) followed by a one-qubit gate on qubit q . (E) Circuit (D) followed by two-qubit gate between qubits q'' and $q'' + 1$.

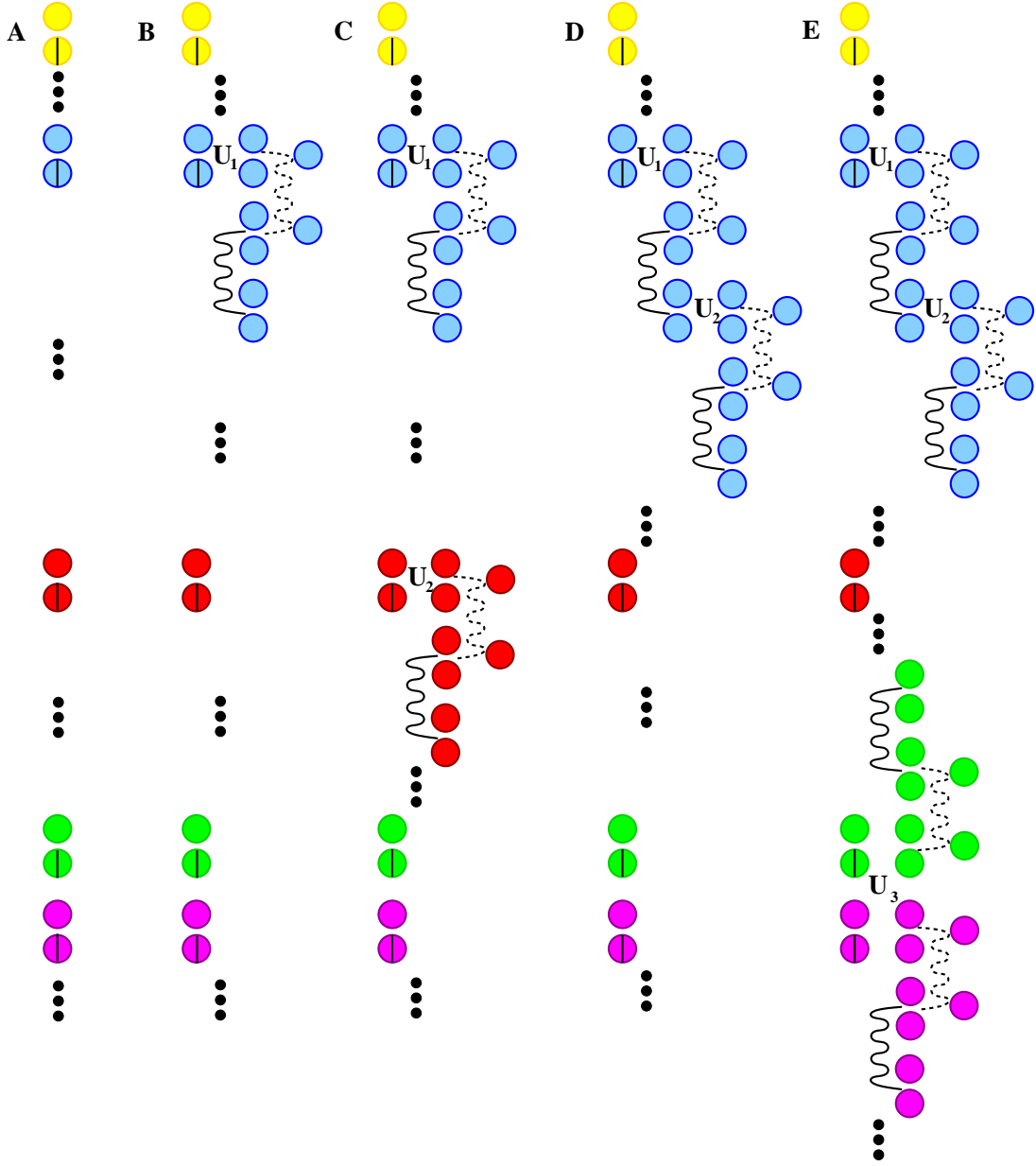


FIG. 2. History-state Hamiltonians corresponding to gates approach circuits in Fig. 1. (A) A pair of circles represents the 2 states of a qubit Hilbert space, like 2 quantum dots or 2 superconducting islands of a charge-based qubit. Vertical black lines represent the energy penalty terms in $\mathcal{H}_{Initialize}$. (B) The $2 \otimes 3 \otimes 5$ Hilbert space of qubit q is depicted with the 5 dimensional part above the 3 dimensional part above the 2 dimensional part. U_1 , the solid ripple, and the dashed ripple represent eq. (1), (3), and (4) respectively. (E) The green $2 \otimes 3 \otimes 5$ dimensional space is oriented with the 5 dimensional part on the bottom to allow a local U_3 interaction.

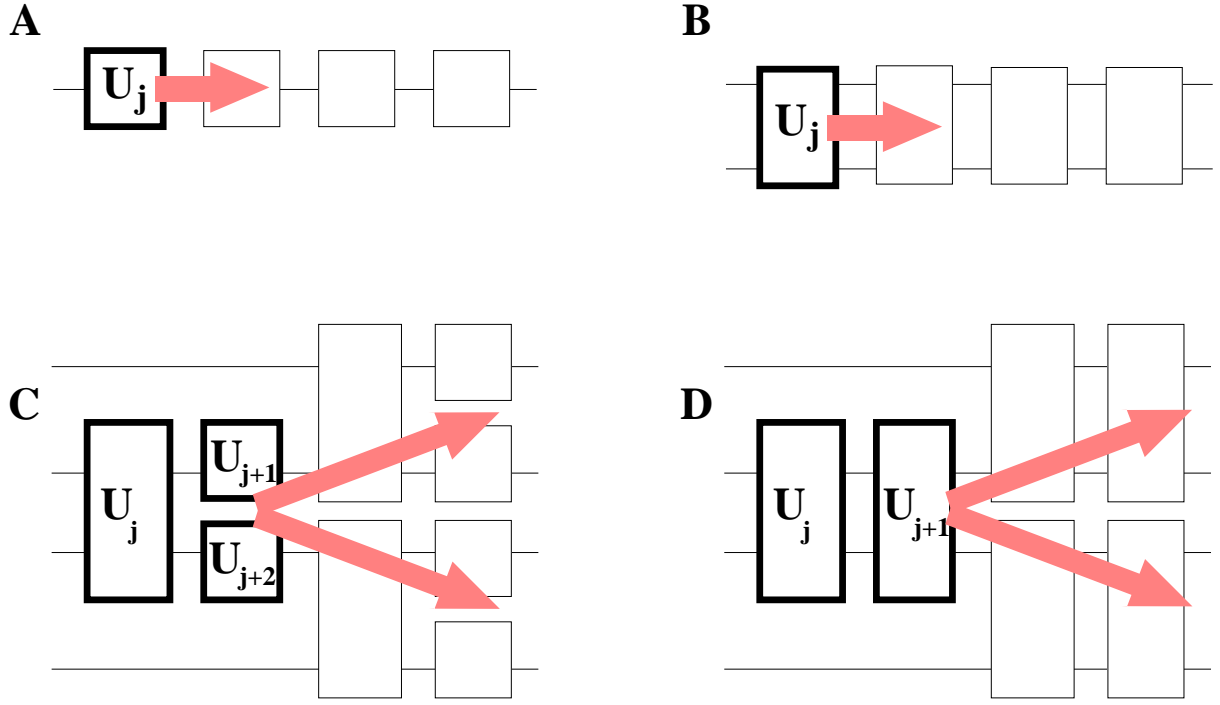


FIG. 3. Translationally invariant circuit geometries. In each case, the red arrow(s) indicate lattice vector(s). (A) One-dimensional chain of one-qubit gates. (B) One-dimensional chain of two-qubit gates. (C) Two-dimensional lattice in which unit cell contains a two-qubit gate and 2 one-qubit gates. (D) Two-dimensional lattice in which unit cell contains 2 two-qubit gates.

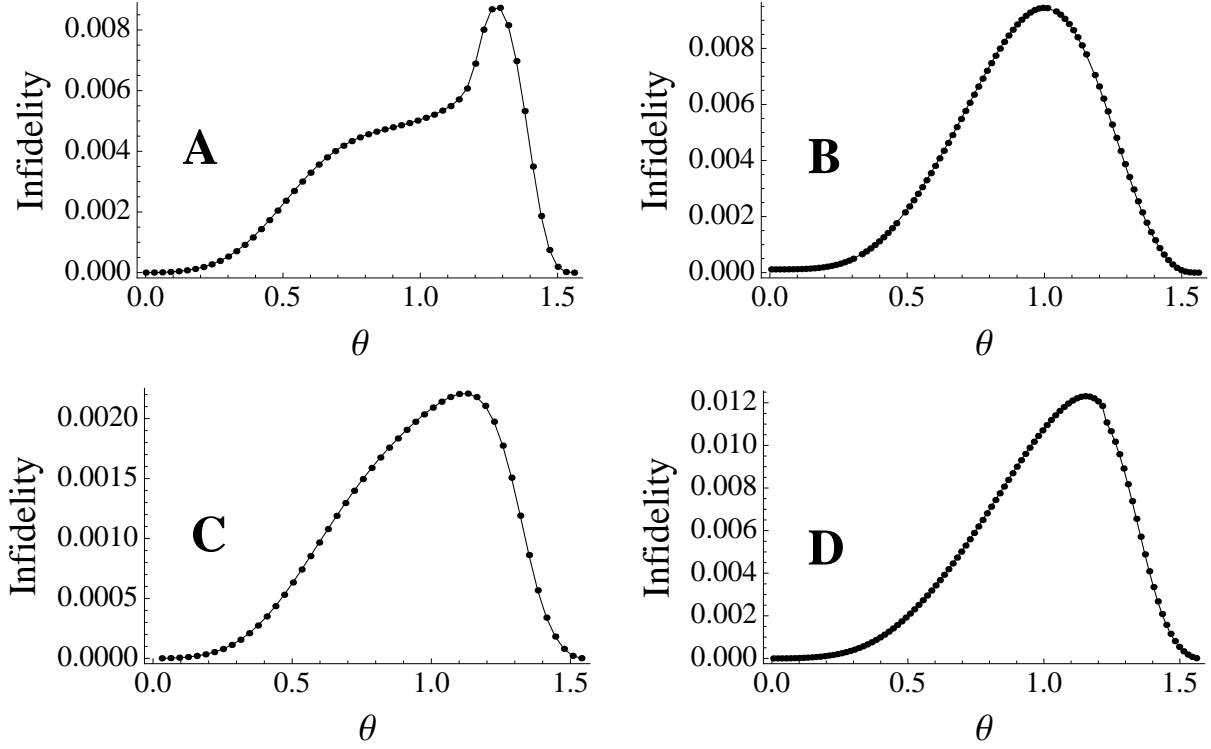


FIG. 4. Infidelity of spin-wave approximation for each of the lattices in Fig. 3. In each case, we compare $|\text{one gate}\rangle$ to $|\text{several gates}\rangle$. Here, $|\text{one gate}\rangle$ is a coherent superposition of states each of which describes an excitation affecting a single gate. The state $|\text{several gates}\rangle$ does not restrict the excitation to a single gate but instead to (A) 4 one-qubit gates, (B) 2 two-qubit gates, (C) a unit cell of the lattice, comprised of a two-qubit gate and 2 one-qubit gates, and (D) a unit cell of the lattice, comprised of 2 two-qubit gates.

APPENDIX

TELEPORTATION

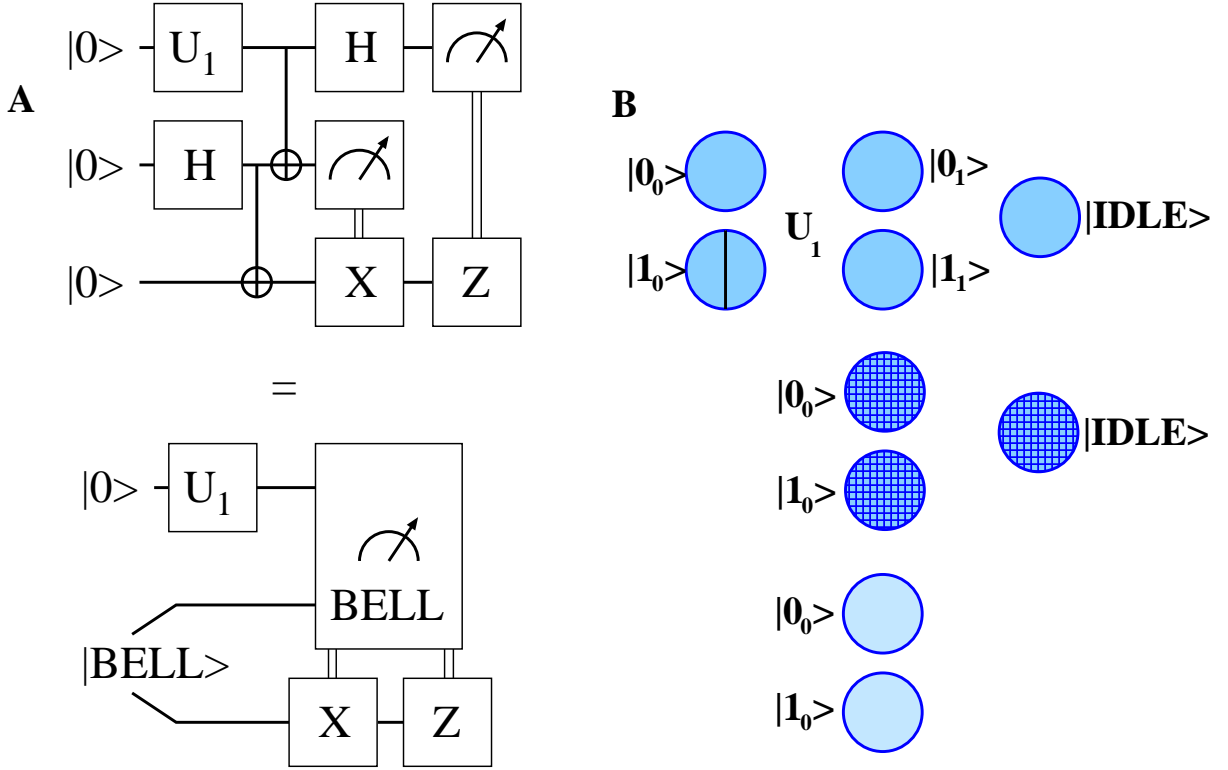


FIG. A1. (A) Quantum teleportation [27], and (B) history-state teleportation. The $2 \otimes 3 \otimes 5$ dimensional Hilbert space of the history-state version is emphasized here with the 5 dimensional part above the 3 dimensional part above the 2 dimensional part; the Hamiltonian is emphasized in Fig. 2B.

HISTORY-STATE HAMILTONIAN TWO-QUBIT GATES

We complete the discussion (iv) of our history-state construction by detailing the case of two-qubit gates. To append a two-qubit gate to the system, we extended the final 2 dimensional Hilbert space of each participating qubit into a $2 \otimes 3 \otimes 5$ Hilbert space; assuming a two-qubit gate between adjacent qubits q'' and $q''+1$, the extension is from a $2 \otimes 2$ dimensional space to a $2 \otimes 3 \otimes 5 \otimes 2 \otimes 3 \otimes 5$

dimensional space. We add to the history-state Hamiltonian \mathcal{H} a term $I \otimes \dots \otimes I \otimes H_{Two-qubit\ gate}^{U_j} \otimes I \otimes \dots \otimes I$. It is comprised of a sum $H_{Two-qubit\ gate}^{U_j} = H^{U_j} + H_{Create\ pairs} \otimes I \otimes I \otimes I + I \otimes I \otimes I \otimes H_{Create\ pairs} \otimes I + I \otimes H_{Bell\ projection}(\theta) \otimes I \otimes I \otimes I + I \otimes I \otimes I \otimes H_{Bell\ projection}(\theta)$. Here, $H_{Create\ pairs}$ and $H_{Bell\ projection}(\theta)$ are familiar from equations (3) and (4). The new ingredient is H^{U_j} . In analogy to case of a one-qubit gate, for the two-qubit gate U_j with matrix elements $\langle b| \otimes \langle B| U_j |b'\rangle \otimes |B'\rangle$, we define $\mathcal{U}_j = \sum_{b,b',B,B',s=0,1} I \otimes I \otimes |b_s\rangle \otimes I \otimes I \otimes |B_s\rangle \langle b| \otimes \langle B| U_j |b'\rangle \otimes |B'\rangle I \otimes I \otimes \langle b'_s| \otimes I \otimes I \otimes \langle B'_s|$. In terms of \mathcal{U}_j ,

$$\begin{aligned} H^{U_j} = & \epsilon \sum_{b,B} [(I \otimes I \otimes |b_0\rangle \otimes I \otimes I \otimes |B_0\rangle) - \mathcal{U}_j (I \otimes I \otimes |b_1\rangle \otimes I \otimes I \otimes |B_1\rangle)] \\ & \left[(I \otimes I \otimes \langle b_0| \otimes I \otimes I \otimes \langle B_0|) - (I \otimes I \otimes \langle b_1| \otimes I \otimes I \otimes \langle B_1|) \mathcal{U}_j^\dagger \right] / 2 \\ & + I \otimes I \otimes |b_0\rangle \langle b_0| \otimes I \otimes I \otimes (|B_1\rangle \langle B_1| + |\text{IDLE}\rangle \langle \text{IDLE}|) \\ & + I \otimes I \otimes (|b_1\rangle \langle b_1| + |\text{IDLE}\rangle \langle \text{IDLE}|) \otimes I \otimes I \otimes |B_0\rangle \langle B_0| \end{aligned} \quad (\text{A1})$$

The first two lines are exactly analogous to the single-qubit gate case (1), despite superficial complexity resulting from the tensor product notation. Both qubits move together from stage 0 to stage 1, undergoing the gate U_j . The next lines impose an energy penalty if either qubit attempts to traverse the gate alone. For the example of a controlled-phase gate, $U_j = U_{CZ}$, we have

$$\begin{aligned} H^{U_{CZ}} = & \epsilon \sum_{b,B} I \otimes I \otimes |b_0\rangle \langle b_0| \otimes I \otimes I \otimes |B_0\rangle \langle B_0| + I \otimes I \otimes |b_1\rangle \langle b_1| \otimes I \otimes I \otimes |B_1\rangle \langle B_1| \\ & + (-1)^{bB} I \otimes I \otimes |b_1\rangle \langle b_0| \otimes I \otimes I \otimes |B_1\rangle \langle B_0| \\ & + (-1)^{bB} I \otimes I \otimes |b_0\rangle \langle b_1| \otimes I \otimes I \otimes |B_0\rangle \langle B_1| \\ & + I \otimes I \otimes |b_0\rangle \langle b_0| \otimes I \otimes I \otimes (|B_1\rangle \langle B_1| + |\text{IDLE}\rangle \langle \text{IDLE}|) \\ & + I \otimes I \otimes (|b_1\rangle \langle b_1| + |\text{IDLE}\rangle \langle \text{IDLE}|) \otimes I \otimes I \otimes |B_0\rangle \langle B_0|. \end{aligned} \quad (\text{A2})$$

There are four degenerate ground states $|\psi_0^{U_j}(b, B)\rangle$ of $H_{Two-qubit\ gate}^{U_j}$, corresponding to four possible inputs $b = 0$ or 1 , $B = 0$ or 1 . For the case $U_j = U_{CZ}$, we have

$$\begin{aligned} |\psi_0^{U_{CZ}}(b, B)\rangle = & \frac{1}{\sqrt{32 \cos^4 \theta + 8 \cos^2 \theta \sin^2 \theta + \sin^4 \theta}} \\ & \left(4 \cos^2 \theta \frac{1}{\sqrt{2}} (|0_0\rangle |0_0\rangle + |1_0\rangle |1_0\rangle) |b_0\rangle \frac{1}{\sqrt{2}} (|0_0\rangle |0_0\rangle + |1_0\rangle |1_0\rangle) |B_0\rangle \right. \end{aligned}$$

$$\begin{aligned}
& +4 \cos^2 \theta (-1)^{bB} \frac{1}{\sqrt{2}} (|0_0\rangle |0_0\rangle + |1_0\rangle |1_0\rangle) |b_1\rangle \frac{1}{\sqrt{2}} (|0_0\rangle |0_0\rangle + |1_0\rangle |1_0\rangle) |B_1\rangle \\
& +2 \cos \theta \sin \theta (-1)^{bB} |b_0\rangle |\text{IDLE}\rangle |\text{IDLE}\rangle \frac{1}{\sqrt{2}} (|0_0\rangle |0_0\rangle + |1_0\rangle |1_0\rangle) |B_1\rangle \\
& +2 \cos \theta \sin \theta \frac{1}{\sqrt{2}} (|0_0\rangle |0_0\rangle + |1_0\rangle |1_0\rangle) (-1)^{bB} |b_1\rangle |B_0\rangle |\text{IDLE}\rangle |\text{IDLE}\rangle \\
& +\sin^2 \theta (-1)^{bB} |b_0\rangle |\text{IDLE}\rangle |\text{IDLE}\rangle |B_0\rangle |\text{IDLE}\rangle |\text{IDLE}\rangle \Big). \tag{A3}
\end{aligned}$$

The first two lines correspond to the action of the U_{CZ} in the standard gate model; the initial state $|b\rangle |B\rangle$ gets carried to $(-1)^{bB} |b\rangle |B\rangle$. In the first two lines, Bell pairs stand ready for the coming teleportations. The next three lines describe teleportation via Bell-basis projection of one pair of qubits (referenced by b in (A1)-(A3) and residing in the $3 \otimes 5$ part of the green $2 \otimes 3 \otimes 5$ space in Fig. 2E), the other pair of qubits (referenced by B in (A1)-(A3) and residing in the $3 \otimes 5$ part of the magenta $2 \otimes 3 \otimes 5$ space in Fig. 2E), or of both pairs of qubits, respectively.

Continuing in analogy to the one-qubit gate, we define the gate operator

$$\hat{g}_0^{U_j} = \sum_{b,B} |\psi_0^{U_j}(b, B)\rangle \langle b_0| \otimes \langle B_0|, \tag{A4}$$

which is a map from a $2 \otimes 2$ dimensional Hilbert space to a $2 \otimes 3 \otimes 5 \otimes 2 \otimes 3 \otimes 5$ dimensional Hilbert space. Tracing over both $3 \otimes 5$ parts of this Hilbert space, we have a superoperator,

$$\begin{aligned}
& \hat{g}_0^{U_j}(\rho) \equiv \text{Tr}_{3 \otimes 5} \text{Tr}_{3 \otimes 5} \hat{g}_0^{U_j} \rho \hat{g}_0^{U_j, \dagger} \\
& = \left(\sin^4 \theta U_j \rho U_j^\dagger \right. \\
& + 4 \cos^2 \theta \sin^2 \theta \frac{|0_0\rangle \langle 0_0| + |1_0\rangle \langle 1_0|}{2} \otimes \text{Tr}_b U_j \rho U_j^\dagger + 4 \cos^2 \theta \sin^2 \theta \text{Tr}_B U_j \rho U_j^\dagger \otimes \frac{|0_0\rangle \langle 0_0| + |1_0\rangle \langle 1_0|}{2} \\
& \left. + 32 \cos^4 \theta \frac{|0_0\rangle \langle 0_0| + |1_0\rangle \langle 1_0|}{2} \otimes \frac{|0_0\rangle \langle 0_0| + |1_0\rangle \langle 1_0|}{2} \right) / (32 \cos^4 \theta + 8 \cos^2 \theta \sin^2 \theta + \sin^4 \theta).
\end{aligned}$$

Here, ρ is a density operator defined on a $2 \otimes 2$ dimensional Hilbert space; $\text{Tr}_b U_j \rho U_j^\dagger$ is obtained by tracing over the first 2 dimensional Hilbert space and $\text{Tr}_B U_j \rho U_j^\dagger$ is obtained by tracing over the second 2 dimensional Hilbert space. This superoperator corresponds to the application of U_j followed by depolarizing channels acting on both qubits. It affects the correct operation U_j with probability $1 - p_0^{U_j} = \sin^4 \theta / (32 \cos^4 \theta + 8 \cos^2 \theta \sin^2 \theta + \sin^4 \theta)$, so that $p_0^{U_j} = (32 \cos^4 \theta + 8 \cos^2 \theta \sin^2 \theta) / (32 \cos^4 \theta + 8 \cos^2 \theta \sin^2 \theta + \sin^4 \theta)$.

PROOF OF FAULT TOLERANCE AGAINST LOCAL STOCHASTIC EXCITATIONS

The fault-tolerance of the gate approach circuit \mathbf{C} relies on having sufficiently many qubits unaffected by noise at any given time. This is guaranteed by the local stochastic fault model probability condition: a fault path, chosen at random, incorporates faults at a specific set of L locations in \mathbf{C} with probability no greater than \bar{p}^L . It does not compromise the fault-tolerance of \mathbf{C} if the small number of qubits that are affected by the noise suffer essentially arbitrary mistreatment.

We state 3 specific fault models that the gate approach circuit \mathbf{C} can tolerate.

- (1) Consider an independent stochastic fault model. Let $g_0^{U_i}$ be a trace-preserving, completely-positive superoperator that applies gate U_i and follows it by a depolarizing channel of probability $p_0^{U_j}$. Let g_0^{Init} be the one-qubit identity superoperator. Suppose executing \mathbf{C} produces $\rho_R = g_0^{U_G}(\dots g_0^{U_1}(g_0^{Init} \dots g_0^{Init}(|0\rangle\langle 0|^{\otimes Q}) \dots) \dots)$. Then the fault-tolerance of \mathbf{C} implies $\text{Tr } R \rho_R / \text{Tr } \rho_R \sim O(1)$ provided that $p_0 = \max_j p_0^{U_j}$ is much less than p .
- (2) Suppose that the faults afflicting \mathbf{C} are non-Markovian, arising by interaction with a shared bath. That is, suppose bath qudits are present that can always undergo gates with one another but undergo gates with the system qubits only at locations for which $F(\ell) \neq 0$. $F(\ell)$ is determined probabilistically in accordance with the local stochastic fault model. Assuming there is no quantum inference among different fault paths, the fault-tolerance of \mathbf{C} still implies $\text{Tr } R \rho_R / \text{Tr } \rho_R \sim O(1)$ provided \bar{p} is much less than p . A proof appears in section 10 of [25] – although the local stochastic fault model probability condition differs slightly from probability condition (10.1) in [25], it leads to the same conclusion (10.6) directly from (10.4). The proof makes no reference to the final state of the bath. Every branch of the bath's final state is consistent with fault-tolerant execution of \mathbf{C} ; we can perform measurements on the bath and postselect for certain outcomes. As long as we maintain the local stochastic fault model probabilities, we still have $\text{Tr } R \rho_R / \text{Tr } \rho_R \sim O(1)$ provided \bar{p} is much less than p .
- (3) We can combine fault models (1) and (2). Suppose that, in addition to the faults that occur by interaction with a shared bath when $F(\ell) \neq 0$, independent stochastic faults occur at all

other locations with probability no greater than p_0 . Thus, the total fault path $F_{tot}(\ell) = F(\ell)$ if $F(\ell) \neq 0$ and $F_{tot}(\ell) \neq 0$ with probability at most p_0 if $F(\ell) = 0$. We still have $\text{Tr } R \rho_R / \text{Tr } \rho_R \sim O(1)$ provided $p_0 + \bar{p}$ is much less than p .

The fault-tolerance properties of the gate approach circuit \mathbf{C} have implications for \mathbf{C} 's history-state equivalent. Consider the local stochastic excitation model described in the main text. First assume the case in which $F(\ell) = 0$ except at the location $\ell = i$ of a single one-qubit gate U_i . We defined the gate operator $\hat{g}_0^{U_i}$ in terms of the ground states (2) annihilated by the Hamiltonian $H_{\text{One-qubit gate}}^{U_i}$. We now define the excited gate operator $\hat{g}_{F(i)}^i \equiv |\psi_{F(i)}^i(0)\rangle \langle 0_0| + |\psi_{F(i)}^i(1)\rangle \langle 1_0|$ in terms of states $|\psi_{F(i)}^i(0)\rangle$ and $|\psi_{F(i)}^i(1)\rangle$ supported on the same $2 \otimes 3 \otimes 5$ dimensional Hilbert space as the ground states (2). These states can depend upon the fault type $F(i)$ and can explicitly depend on the location i (rather than depending merely parametrically on i through the gate U_i like the ground states (2)). Given our choice of $F(\ell)$, the most general form of $|\Psi_F\rangle$ is $|\Psi_F\rangle = \hat{g}_0^{U_G} \dots \hat{g}_0^{U_{i+1}} \hat{g}_{F(i)}^i \hat{g}_0^{U_{i-1}} \dots \hat{g}_0^{Init} |0_0\rangle^{\otimes Q}$. Now, we trace out all of the $3 \otimes 5$ dimensional Hilbert spaces from the density matrix to obtain $\rho_{R,F} = g_0^{U_G} \left(\dots g_0^{U_{i+1}} \left(\text{Tr}_{3 \otimes 5} [\hat{g}_{F(i)}^i g_0^{U_{i-1}} \left(\dots g_0^{Init} (|0\rangle \langle 0|^{\otimes Q}) \dots \right) \hat{g}_{F(i)}^{i\dagger}] \right) \dots \right)$. After normalizing $|\Psi_F\rangle$, the true density matrix is $\rho_{R,F} / \text{Tr } \rho_{R,F}$.

We demonstrate that the same density matrix could be obtained by executing \mathbf{C} in the gate approach, with the same $F(\ell)$, under fault model (3). Imagine executing the gates of \mathbf{C} , and at location i encountering a $(3 \otimes 5)$ dimensional bath initially in some given state such as $(|0_0\rangle \otimes |0_0\rangle)$. Just before location i , the density matrix is $g_0^{U_{i-1}} \left(\dots g_0^{Init} (|0\rangle \langle 0|^{\otimes Q}) \dots \right)$. At location i , suppose a system-bath unitary gate U_{SB} acts on the qubit \otimes bath Hilbert space, carrying $|0_0\rangle \otimes (|0_0\rangle \otimes |0_0\rangle)$ to $\frac{1}{\sqrt{2}} |0_0\rangle \otimes (|0_0\rangle \otimes |0_0\rangle) + \frac{1}{\sqrt{2}} |1_0\rangle \otimes (|1_0\rangle \otimes |0_0\rangle)$ and $|1_0\rangle \otimes (|0_0\rangle \otimes |0_0\rangle)$ to $\frac{1}{\sqrt{2}} |0_0\rangle \otimes (|0_0\rangle \otimes |1_0\rangle) + \frac{1}{\sqrt{2}} |1_0\rangle \otimes (|1_0\rangle \otimes |1_0\rangle)$, so that the bath stores the initial and final states of the qubit. Defining $|\psi_{F(i)}^i(0)\rangle = |0_0\rangle \otimes |\psi_{F(i)}^i(0,0)\rangle + |1_0\rangle \otimes |\psi_{F(i)}^i(1,0)\rangle$ and $|\psi_{F(i)}^i(1)\rangle = |0_0\rangle \otimes |\psi_{F(i)}^i(0,1)\rangle + |1_0\rangle \otimes |\psi_{F(i)}^i(1,1)\rangle$, we apply a transformation on the bath state alone to carry $|b'_0\rangle \otimes |b_0\rangle / \sqrt{2}$ to $|\psi_{F(i)}^i(b', b)\rangle$. Since the $|\psi_{F(i)}^i(b', b)\rangle$ are not necessarily orthogonal vectors with norm $1/\sqrt{2}$, this transformation may require measurement and postselection rather than just unitary gates (see below). After this non-unitary transformation, the qubit-bath density matrix

at location i is $\hat{g}_{F(i)}^i g_0^{U_{i-1}} \left(\dots g_0^{Init} \left(|0\rangle \langle 0|^{\otimes Q} \right) \dots \right) \hat{g}_{F(i)}^{i\dagger}$. Continuing until the end of the circuit yields $\rho_{R,F}/\text{Tr } \rho_{R,F}$ once we trace out the bath degrees of freedom.

This analysis generalizes to arbitrary fault paths with one complication. Consider the case in which $F(\ell) \neq 0$ at 2 locations i, j at which one-qubit gates act on different qubits. In this case, the most general form of $|\Psi_F\rangle$ is not $\hat{g}_0^{U_G} \dots \hat{g}_0^{U_{j+1}} \hat{g}_{F(j)}^j \hat{g}_0^{U_{j-1}} \dots \hat{g}_0^{U_{i+1}} \hat{g}_{F(i)}^i \hat{g}_0^{U_{i-1}} \dots \hat{g}_0^{Init} |0_0\rangle^{\otimes Q}$ since the degrees-of-freedom at location i can be entangled with the degrees-of-freedom at location j . Instead, the most general (unnormalized) form is a sum

$$\begin{aligned} |\Psi_F\rangle = & \hat{g}_0^{U_G} \dots \hat{g}_0^{U_{j+1}} \hat{g}_{F(j)}^j \hat{g}_0^{U_{j-1}} \dots \hat{g}_0^{U_{i+1}} \hat{g}_{F(i)}^i \hat{g}_0^{U_{i-1}} \dots \hat{g}_0^{Init} |0_0\rangle^{\otimes Q} \\ & + \hat{g}_0^{U_G} \dots \hat{g}_0^{U_{j+1}} \hat{g}_{F(j)}^{j'} \hat{g}_0^{U_{j-1}} \dots \hat{g}_0^{U_{i+1}} \hat{g}_{F(i)}^{i'} \hat{g}_0^{U_{i-1}} \dots \hat{g}_0^{Init} |0_0\rangle^{\otimes Q} \\ & + \hat{g}_0^{U_G} \dots \hat{g}_0^{U_{j+1}} \hat{g}_{F(j)}^{j''} \hat{g}_0^{U_{j-1}} \dots \hat{g}_0^{U_{i+1}} \hat{g}_{F(i)}^{i''} \hat{g}_0^{U_{i-1}} \dots \hat{g}_0^{Init} |0_0\rangle^{\otimes Q} + \dots \end{aligned}$$

where we have defined gate operators $\hat{g}_{F(j)}^j$, $\hat{g}_{F(j)}^{j'}$, $\hat{g}_{F(j)}^{j''}$, $\hat{g}_{F(i)}^{i'}$, and $\hat{g}_{F(i)}^{i''}$ in analogy to $\hat{g}_{F(i)}^i$ using states $|\psi_{F(j)}^j(b)\rangle$, $|\psi_{F(j)}^{j'}(b)\rangle$, $|\psi_{F(j)}^{j''}(b)\rangle$, $|\psi_{F(i)}^{i'}(b)\rangle$, and $|\psi_{F(i)}^{i''}(b)\rangle$. Even with this additional complexity, $\rho_{R,F}/\text{Tr } \rho_{R,F}$ of the normalized $|\Psi_F\rangle$ could have been obtained by executing **C** in the gate approach, with the same $F(\ell)$, under fault model (3). To see this, define and decompose $|\psi_{F(j),F(i)}^{j,i}(b_0, B_0)\rangle = |\psi_{F(j)}^j(b)\rangle |\psi_{F(i)}^i(B)\rangle + |\psi_{F(j)}^{j'}(b)\rangle |\psi_{F(i)}^{i'}(B)\rangle + |\psi_{F(j)}^{j''}(b)\rangle |\psi_{F(i)}^{i''}(B)\rangle + \dots = \sum_{b', B'} |b'\rangle \otimes |B'\rangle \otimes |\psi_{F(j),F(i)}^{j,i}(b', B'; b_0, B_0)\rangle$. Execute **C** and, at locations i and j , suppose U_{SB} gets applied to the the qubit and a bath at each location to record the initial and final qubit states into the bath. Just before **C** is finished executing, a non-unitary transformation on the shared bath carries $|b'\rangle \otimes |b_0\rangle / \sqrt{2} \otimes |B'\rangle \otimes |B_0\rangle / \sqrt{2}$ to $|\psi_{F(j),F(i)}^{j,i}(b', B'; b_0, B_0)\rangle$. After this non-unitary transformation, $\rho_{R,F}/\text{Tr } \rho_{R,F}$ is obtained once we trace out the bath degrees of freedom.

The same argument generalizes immediately to arbitrary fault paths: we define and decompose $|\psi_{F(j),F(i),\dots}^{j,i,\dots}(b_0, B_0, \dots)\rangle$, apply U_{SB} at each fault location and then use a non-unitary transformation on the overall bath as the final step in **C**. It follows in general that the history-state output, $\text{Tr } R \rho_R = \sum_F p_F \text{Tr } R \rho_{R,F}/\text{Tr } \rho_{R,F}$ assuming local stochastic excitation, could have been produced by executing **C** with fault model (3).

To complete the proof, we just need to detail the non-unitary transformation used above. Let $|\alpha_i\rangle$, $i = 1, \dots, A$, comprise a set of orthonormal states $\langle \alpha_i | \alpha_j \rangle = \delta_{i,j}$. Let $|\beta_i\rangle$, $i = 1, \dots, A$, comprise a set of states that are not necessarily orthogonal or normalized. We will present a

transformation $|\alpha_i\rangle \rightarrow |\beta_i\rangle$ that employs unitary gates, measurement, and postselection. As emphasized above in our statement of fault-model (2), it is permissible to apply these operations to the bath without compromising the fault-tolerance of the gate approach circuit C. Our priority is simplicity; we make no attempt to maximum the success rate or to minimize the number of ancilla required.

We will make use of an auxilliary orthonormal set of ancilla states $|\gamma_i\rangle$, $\langle\gamma_i|\gamma_j\rangle = \delta_{i,j}$ with $i = 1, \dots, A + 1$. To affect the desired mapping, begin with a unitary gate carrying $|\alpha_i\rangle \otimes |\gamma_1\rangle \rightarrow |\beta_i\rangle \otimes (|\gamma_1\rangle - \sum_{j=2}^i c_j |\gamma_j\rangle + |\gamma_{i+1}\rangle) / \sqrt{2 + \sum_{j=2}^i c_j^2}$. Here the c_i coefficients are defined by $c_1 = 1$ and $c_j \equiv \sum_{k=1}^{j-1} c_k^2$ for $j > 1$. To show this mapping is unitary, take $i' > i$ and compute $(\langle\gamma_1| - \sum_{j=2}^i c_j \langle\gamma_j| + \langle\gamma_{i+1}|) \cdot (|\gamma_1\rangle - \sum_{j=2}^{i'} c_j |\gamma_j\rangle + |\gamma_{i'+1}\rangle) = 1 + \sum_{j=2}^i c_j^2 - c_{i+1} = 1 + \sum_{j=2}^i c_j^2 - \sum_{k=1}^i c_k^2 = 1 - c_1^2 = 0$. Thus, orthonormal states have been mapped to orthonormal states.

After the unitary gate, measure the γ register. If the result is $|\gamma_1\rangle$, then the transformation $|\alpha_i\rangle \rightarrow |\beta_i\rangle$ is complete. For any other result, start again. These failed attempts are irrelevant if we postselect our output based on successfully measuring $|\gamma_1\rangle$.

FABRICATION FAULTS

Consider a model of fabrication faults in which the actual Hamiltonian \mathcal{H}' differs from \mathcal{H} at locations satisfying $F(\ell) \neq 0$. Here, $F(\ell)$ satisfies the probability condition borrowed from the local stochastic fault model. Assume for simplicity that the ground state of \mathcal{H}' still has zero energy. In that case, its ground-state is just a locally excited state of \mathcal{H} . It follows that $\text{Tr } R \rho_R \sim O(1)$ provided $p_0 + \bar{p}$ is much less than p .

We see no reason that allowing a different ground state energy of \mathcal{H}' will compromise fault tolerance, but we have not attempted a careful argument. One can show fault tolerance against a leakage model. In general, we expect the history-state construction to be fault tolerant against various models combining fabrication faults and excitation faults.

INFIDELITY SIMULATIONS

In the case of the single-qubit gate chain (Fig. 3A), the infidelity calculation is based upon a comparison of the state $|\text{one gate}\rangle$ introduced in the main text and a state $|\text{several gates}\rangle$. Because it takes the unit cell of the chain to consist of 4 single-qubit gates, we take $|\text{several gates}\rangle = \sum_j e^{i4kj} \dots \hat{g}_0^{U_{4(j+1)+3}, \dots, U_{4(j+1)}} \hat{g}_f^{U_{4j+3}, \dots, U_{4j}} \hat{g}_0^{U_{4(j-1)+3}, \dots, U_{4(j-1)}} \dots |0\rangle$. Here, the gate operators $\hat{g}_0^{U_{4j+3}, U_{4j+2}, U_{4j+1}, U_{4j}} = \hat{g}_0^{U_{4j+3}} \hat{g}_0^{U_{4j+2}} \hat{g}_0^{U_{4j+1}} \hat{g}_0^{U_{4j}}$ and $\hat{g}_f^{U_{4j+3}, U_{4j+2}, U_{4j+1}, U_{4j}} = \sum_b |\psi_f^{U_{4j+3}, U_{4j+2}, U_{4j+1}, U_{4j}}(b)\rangle \langle b|$ are each maps from a 2 dimensional Hilbert space to a $2 \otimes 3 \otimes 5 \otimes 3 \otimes 5 \otimes 3 \otimes 5 \otimes 3 \otimes 5$ dimensional Hilbert space. To compare $|\text{one gate}\rangle$ and $|\text{several gates}\rangle$, we define the infidelity as $1 - (1/2) \sum_b \langle b| (\hat{g}_f^{U_{4j+3}, U_{4j+2}, U_{4j+1}, U_{4j}})^\dagger (\hat{g}_0^{U_{4j+3}} \hat{g}_0^{U_{4j+2}} \hat{g}_0^{U_{4j+1}} \hat{g}_f^{U_{4j}} + \hat{g}_0^{U_{4j+3}} \hat{g}_0^{U_{4j+2}} \hat{g}_f^{U_{4j+1}} \hat{g}_0^{U_{4j}} + \hat{g}_0^{U_{4j+3}} \hat{g}_f^{U_{4j+2}} \hat{g}_0^{U_{4j+1}} \hat{g}_0^{U_{4j}} + \hat{g}_f^{U_{4j+3}} \hat{g}_0^{U_{4j+2}} \hat{g}_0^{U_{4j+1}} \hat{g}_0^{U_{4j}}) |b\rangle / 2$. This infidelity is plotted in Fig. 4A.

For the chain of two-qubit gates as shown in Fig. 3B, we compare a simulation with one gate per unit cell to a simulation with two gates per unit cell. For a two-dimensional lattice with one two-qubit gate U_j and two one-qubit gates U_{j+1} and U_{j+2} per unit cell, Fig. 3C, we determine whether the solution $\hat{g}_f^{U_j, U_{j+1}, U_{j+2}}$ can be decomposed into an excitation on each gate $\hat{g}_f^{U_{j+2}} \hat{g}_0^{U_{j+1}} \hat{g}_0^{U_j} + \hat{g}_0^{U_{j+2}} \hat{g}_f^{U_{j+1}} \hat{g}_0^{U_j} + \hat{g}_0^{U_{j+2}} \hat{g}_0^{U_{j+1}} \hat{g}_f^{U_j}$. For a two-dimensional lattice with 2 two-qubit gates U_j and U_{j+1} , Fig. 3D, we determine whether the solution $\hat{g}_f^{U_j, U_{j+1}}$ can be decomposed into an excitation on each gate $\hat{g}_f^{U_{j+1}} \hat{g}_0^{U_j} + \hat{g}_0^{U_{j+1}} \hat{g}_f^{U_j}$. The results appear in Fig. 4B, C, and D.

In the region of computational interest $\theta \sim \pi/2$, we can get a sense of the form of the spin-wave excitations for a general history-state Hamiltonian. The ground state $|\Psi\rangle$ has all of its $3 \otimes 5$ dimensional spaces in $|IDLE\rangle \otimes |IDLE\rangle$ for $\theta \sim \pi/2$. Roughly speaking the spin-wave states replace one $|IDLE\rangle \otimes |IDLE\rangle$ with a hopping Bell pair. There are 3 choices of hopping Bell pair: $(|0_0\rangle |0_1\rangle - |1_0\rangle |1_1\rangle)/\sqrt{2}$, $(|0_0\rangle |1_1\rangle + |1_0\rangle |0_1\rangle)/\sqrt{2}$, or $(|0_0\rangle |1_1\rangle - |1_0\rangle |0_1\rangle)/\sqrt{2}$. The other Bell pair, $(|0_0\rangle |0_1\rangle + |1_0\rangle |1_1\rangle)/\sqrt{2}$, is not used to form a low-energy spin wave because Hamiltonian (4) pushes up its energy.

GENERAL BOSON STATE

We write a general state $|n(k, f)\rangle = \sum_F \chi_{n(k, f)}(F) |\Psi_F\rangle$ under the assumption of non-interacting bosons. For the case of a single boson, $\chi(F) = \xi_{k, f}(G) \delta_{F, (f, 0, \dots, 0)} + \xi_{k, f}(G-1) \delta_{F, (0, f, \dots, 0)} + \dots$ can be rewritten as $\sum_{\ell_{k, f}=1}^G \xi_{k, f}(\ell_{k, f}) \delta_{F(\ell_{k, f}), f} \prod_{\ell' \neq \ell_{k, f}} \delta_{F(\ell'), 0}$, where $\ell_{k, f}$ is a location in the circuit. This form is amenable to generalization to a state with a distribution $n(k, f)$ of bosons: $\chi_{n(k, f)}(F, b) = \prod_{k, f} \sum_{\ell_{k, f}^1 \neq \dots \neq \ell_{k, f}^{n(k, f)}} \xi_{k, f}(\ell_{k, f}^1) \delta_{F(\ell_{k, f}^1), f} \dots \xi_{k, f}(\ell_{k, f}^{n(k, f)}) \delta_{F(\ell_{k, f}^{n(k, f)}), f} \prod_{\ell' \neq \ell_{k, f}^j} \delta_{F(\ell'), 0}$.

STATISTICAL MECHANICS OF HISTORY-STATE HAMILTONIAN

Thermal average of $\text{Tr } R \rho_R / \text{Tr } \rho_R$ is $\text{Tr } R e^{-\beta \mathcal{H}} / \text{Tr } e^{-\beta \mathcal{H}}$. At $T = 0$, we know this is the ground state expectation $\langle \Psi | R | \Psi \rangle$, which is $O(1)$ for θ close to $\pi/2$. What happens when $T > 0$? Do the low-lying excited states involve large domains of faults that overwhelm the error-correcting capacity of **C**? The spin-wave bosonic excitations discussed in the main text should not cause this problem unless too many bosons get excited. Physically speaking, we expect the density of bosons over the gates of \mathcal{H} to go as $\sum_{k, f} \langle n(k, f) \rangle / G + Q$ where $\langle n(k, f) \rangle = 1/(e^{\beta E(k, f)} - 1)$ is the average boson occupancy. We assume a density of states of the hopping bosons that goes like $(G + Q)\sqrt{E}$ in 3 spatial dimensions irrespective of the specifics of the circuit. This leads to a boson density that does not depend on $G + Q$ and vanishes with decreasing temperature; if the temperature is low enough, scalable computation should be possible.

To make this physical argument more explicit, we approximate

$$\begin{aligned} \frac{\text{Tr } R \rho_R}{\text{Tr } \rho_R} &= \frac{\text{Tr } R e^{-\beta \mathcal{H}}}{\text{Tr } e^{-\beta \mathcal{H}}} \approx \frac{\sum_{n(k, f)} \langle n(k, f) | R | n(k, f) \rangle e^{-\beta \sum_{k, f} E(k, f) n(k, f)}}{\sum_{n(k, f)} e^{-\beta \sum_{k, f} E(k, f) n(k, f)}} \\ &= \frac{\sum_{n(k, f)} \sum_{F, \bar{F}} \langle \Psi_{\bar{F}} | R | \Psi_{\bar{F}} \rangle \chi_{n(k, f)}(\bar{F}) \chi_{n(k, f)}^*(F) e^{-\beta \sum_{k, f} E(k, f) n(k, f)}}{\sum_{n(k, f)} e^{-\beta \sum_{k, f} E(k, f) n(k, f)}}. \end{aligned} \quad (\text{A5})$$

We expect $\langle \Psi_{\bar{F}} | R | \Psi_{\bar{F}} \rangle$ to be small unless F is close to \bar{F} . To see this, label **C**'s gate locations as $\ell = 1, \dots, G$ and its initialization locations as $\ell = -Q + 1, \dots, 0$; recall that **C** has no measurement locations. We write $|\Psi_{\bar{F}}\rangle = \hat{g}_{\bar{F}(G)}^G \dots \hat{g}_{\bar{F}(-Q+1)}^{-Q+1} |0\rangle^{\otimes Q}$ and $|\Psi_F\rangle = \hat{g}_{F(G)}^G \dots \hat{g}_{F(-Q+1)}^{-Q+1} |0\rangle^{\otimes Q}$.

Then,

$$\langle \Psi_F | R | \Psi_{\bar{F}} \rangle = \text{Tr}_2 \dots \text{Tr}_2 R \text{Tr}_{3 \otimes 5} \dots \text{Tr}_{3 \otimes 5} \hat{g}_{\bar{F}(G)}^G \dots \hat{g}_{\bar{F}(-Q+1)}^{-Q+1} |0\rangle \langle 0|^{\otimes Q} \hat{g}_{F(-Q+1)}^{-Q+1, \dagger} \dots \hat{g}_{F(G)}^{G, \dagger}. \quad (\text{A6})$$

The functions of the form $\text{Tr}_{3 \otimes 5} \hat{g}_{\bar{F}(\ell)}^\ell \rho \hat{g}_{F(\ell)}^{\ell, \dagger}$ that appear here will tend to be small when $\bar{F}(\ell) \neq F(\ell)$. We can prove a bound in the case $\text{Tr}_{3 \otimes 5} \hat{g}_0^{U_\ell} \rho \hat{g}_{F(\ell)}^{\ell, \dagger}$. Suppose that ℓ is the location of a one-qubit gate. Using the fact that $\langle \psi_{F(\ell)}^\ell(b) | \psi_0^{U_\ell}(b') \rangle = 0$ and the form (2), we conclude that $|\langle \psi_{F(\ell)}^\ell(b) | b'_0 \rangle \otimes |\text{IDLE}\rangle \otimes |\text{IDLE}\rangle| \sim O(\cot \theta)$. This implies that $\text{Tr}_{3 \otimes 5} \hat{g}_0^{U_\ell} \rho \hat{g}_{F(\ell)}^{\ell, \dagger} \sim O(\cos^2 \theta) \ll 1$ in the region of computational interest, in which θ is near $\pi/2$. A similar argument works for two-qubit gates. Because the density of faults in the low energy states is low, if $\bar{F}(\ell) \neq F(\ell)$ at some point ℓ , then usually either $F(\ell) = 0$ or $\bar{F}(\ell) = 0$, so the bound on $\text{Tr}_{3 \otimes 5} \hat{g}_0^{U_\ell} \rho \hat{g}_{F(\ell)}^{\ell, \dagger}$ bounds $\langle \Psi_F | R | \Psi_{\bar{F}} \rangle$. This addresses all cases of $\bar{F} \neq F$ except the case in which \bar{F} shares all the same fault locations as F but permutes the fault types among these locations. Unless the permutation of fault types occurs among nearby fault locations, however, the imaginary-time propagator in (A5), $\sum_{n(k,f)} \chi_{n(k,f)}(\bar{F}) \chi_{n(k,f)}^*(F) e^{-\beta \sum_{k,f} E(k,f) n(k,f)} = \langle \Psi_F | e^{-\beta \mathcal{H}} | \Psi_{\bar{F}} \rangle$, should be relatively small. These observations suggest the approximation

$$\frac{\text{Tr } R \rho_R}{\text{Tr } \rho_R} \approx \sum_F \langle \Psi_F | R | \Psi_F \rangle p_F \quad (\text{A7})$$

where

$$p_F = \frac{\langle \Psi_F | e^{-\beta \mathcal{H}} | \Psi_F \rangle}{\text{Tr } e^{-\beta \mathcal{H}}} = \frac{\sum_{n(k,f)} |\chi_{n(k,f)}(F)|^2 e^{-\beta \sum_{k,f} E(k,f) n(k,f)}}{\sum_{n(k,f)} e^{-\beta \sum_{k,f} E(k,f) n(k,f)}}.$$

Given this approximation, we can immediately write (A6) as

$$\langle \Psi_F | R | \Psi_F \rangle = \text{Tr } R g_{F(G)}^G \left(\dots g_{F(-Q+1)}^{-Q+1} \left(|0\rangle \langle 0|^{\otimes Q} \right) \dots \right)$$

in terms of superoperators $g_{F(\ell)}^\ell$. This is the probability of successfully executing **C** in the gate approach, given a fault path F and additional faults occurring with probability p_0 at locations $F(\ell) = 0$.

The more complicated part of (A7) is the probability p_F . We mentioned that $\langle \Psi_F | e^{-\beta \mathcal{H}} | \Psi_F \rangle$ is a propagator for imaginary times $\beta = -it/\hbar$. The temperature, while low compared to the

energy scale ϵ of the Hamiltonian, is much greater than the gap between the ground state and the first excited state. (Otherwise, fault-tolerance against thermal excitations becomes trivial.) This corresponds to a short time t over which the propagator $\langle \Psi_F | e^{-\beta \mathcal{H}} | \Psi_F \rangle$ should be relatively insensitive to the absolute positions of the faults in F relative to the boundaries of the circuit. As a result, the thermal average should not depend upon the exact form of the functions $\xi_{k,f}$ and $\chi_{n(k,f)}$; we expect these functions to be smooth in amplitude but do not need to compute them precisely. Taking the amplitude of $|\chi_{n(k,f)}(F)|^2$ to be evenly distributed over the system, we get $p_F = \sum_{n(k,f)}^F \left(\sum_{k,f}^{G+Q} n(k,f) \right)^{-1} e^{-\beta \sum_{k,f} E(k,f)n(k,f)} / \text{Tr} e^{-\beta \mathcal{H}}$ where $\sum_{n(k,f)}^F$ in the numerator runs only over spin waves consistent with the fault path ($\sum_{\ell} \delta_{F(\ell),f} = \sum_k n(k,f)$ and $\sum_{k,f} n(k,f)$ is the number of faults in F).

To verify consistency with the local stochastic excitation probability constraint on F , choose a set \mathbf{L} comprised of L locations. Let $F(\mathbf{L})$ denote the set of fault paths with faults exactly at \mathbf{L} but with arbitrary fault types: $F(\mathbf{L}) = \{F | F(\ell) \neq 0 \iff \ell \in \mathbf{L}\}$. Then $p_{F(\mathbf{L})} = \sum_{F \in F(\mathbf{L})} p_F$. Our approximation for $|\chi_{n(k,f)}(F)|^2$ implies $p_{F(\mathbf{L})}$ depends upon L but not upon the specific locations in \mathbf{L} . To evaluate $p_{F(\mathbf{L})}$, we note that, for non-interacting bosons, the partition function is obtained by summing geometric series $\text{Tr} e^{-\beta \mathcal{H}} = 1 / \prod_{k,f} (1 - e^{-\beta E(k,f)})$. To sum only terms with L excitations, we introduce a variable x and use our expression for p_F to write

$$p_{F(\mathbf{L})} = \binom{G+Q}{L}^{-1} \frac{1}{L!} \frac{d^L}{dx^L} \prod_{k,f} \frac{(1 - e^{-\beta E(k,f)})}{(1 - x e^{-\beta E(k,f)})} \Big|_{x=0}. \quad (\text{A8})$$

If F is chosen at random, the chances that it will include faults at \mathbf{L} , and possibly other locations too, is $P_{F(\mathbf{L})} \equiv p_{F(\mathbf{L})} + \sum_{\ell' \notin \mathbf{L}} p_{F(\mathbf{L} \cup \{\ell'\})} + \sum_{\ell', \ell'' \notin \mathbf{L}} p_{F(\mathbf{L} \cup \{\ell', \ell''\})} + \dots < \binom{G+Q}{L}^{-1} (1/L!) d^L / dx^L (1 + d/dx + (1/2!) d^2/dx^2 + \dots) \prod_{k,f} (1 - e^{-\beta E(k,f)}) / (1 - x e^{-\beta E(k,f)}) \Big|_{x=0} = \binom{G+Q}{L}^{-1} (1/L!) d^L / dx^L \prod_{k,f} (1 - e^{-\beta E(k,f)}) / (1 - (x+1) e^{-\beta E(k,f)}) \Big|_{x=0}$. Evaluating the derivative, we obtain

$$P_{F(\mathbf{L})} < \binom{G+Q}{L}^{-1} \sum_{(k_1, f_1) \geq (k_2, f_2) \geq \dots (k_L, f_L)} \langle n(k_1, f_1) \rangle \dots \langle n(k_L, f_L) \rangle. \quad (\text{A9})$$

This expression involves averaging over all ways of choosing L locations and, for each way, taking a product to determine the probability that all L locations are faulty (i.e. occupied by a bosonic excitation). Setting $p(T) = \max_L \left[\binom{G+Q}{L}^{-1} \sum_{(k_1, f_1) \geq (k_2, f_2) \geq \dots (k_L, f_L)} \langle n(k_1, f_1) \rangle \dots \langle n(k_L, f_L) \rangle \right]^{1/L}$,

we can bound $P_{F(\mathbf{L})} < p(T)^L$, confirming consistency with the local stochastic excitation model. We estimate $p(T)$ with $\sum_{k,f} \langle n(k, f) \rangle / G + Q$. If there isn't an excitation a given location, there is still some probability of a fault there since $g_0^{U_\ell}$ has a fault probability p_0 . Thus, the probability that the total fault path will include the locations in \mathbf{L} is no greater than $p_0^L + \binom{L}{1} p_0^{L-1} p(T)^1 + \dots = (p_0 + p(T))^L$.

To evaluate $p(T)$, one uses the fact the $E(k, f)$ shows quadratic dispersion, as shown in the following figure and discussed in the main text.

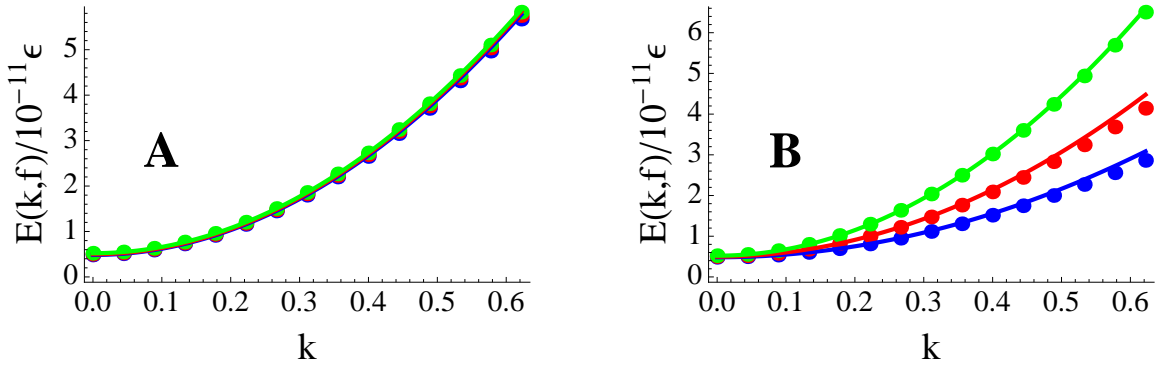


FIG. A2. Spin-wave excitations $E(k, f)$ of lattice depicted in Fig. 3D for lowest-lying modes $f = 1, 2, 3$. Quadratic fits to the numerical data are excellent for both (A) momentum right and (B) momentum up.